

Static potentials for quarkonia at finite temperatures

Owe Philipsen

Institut für Theoretische Physik, Universität Münster, 48149 Münster, Germany

Abstract

We review non-perturbative static potentials commonly used in potential models for quarkonia at finite T . Potentials derived from Polyakov loop correlators are shown to be inappropriate for this purpose. The $q\bar{q}$ free energy is physical but has the wrong spatial decay and perturbative limit. The so-called singlet free energy is gauge dependent and unphysical. An appropriate static real time potential can be defined through a generalisation of pNRQCD to finite T . In perturbation theory, its real part reproduces the Debye-screened potential, its imaginary part accounts for Landau damping. Possibilities for its non-perturbative evaluation are discussed.

Key words: Thermal field theory, Lattice gauge theory, Quark gluon plasma, Quarkonia

PACS: 11.10.Wx, 11.15.Ha, 12.38Gc, 12.38Mh, 12.39Pn

1. Introduction

The properties of quarkonia are believed to provide a useful probe of the QCD plasma at high temperatures, in particular for the quark-hadron transition. This expectation was originally based on a potential model [1], in which the linearly confining potential for zero temperature gets replaced by a Debye-screened potential at high T .

Potential models have a long history for the description of quarkonia at zero temperature. The basic idea is that for heavy quarks of mass M , which move non-relativistically, the binding energy ($E - 2M$) is small compared to M and can be obtained by solving a static Schrödinger equation

$$\left(\frac{\nabla^2}{M} + V(r)\right)\psi = (E - 2M)\psi. \quad (1)$$

$V(r)$ is the (radially symmetric) potential between the static quark anti-quark pair separated by a distance r . Initially $V(r)$ was modelled by the Cornell potential (Coulomb plus linear), more recently non-perturbative lattice data are used as input. The crucial observation is that the Schrödinger equation follows from an effective theory approach.

Starting from QCD, one can use of the scale separation between the heavy quark mass M and the binding energy $E - 2M$, to obtain an effective theory, pNRQCD [2], for the low energy dynamics in the confining potential. In this framework, the static potential appears as a perturbative matching coefficient of the effective theory. Hence, the Schrödinger equation can be improved systematically by computing higher order terms in the scale hierarchy. Note, that a very successful spectroscopy with $\sim 1\%$ accuracy is obtained in this way.

It is tempting to employ this approach also at finite T . Matsui and Satz heuristically used the same equation, but with a Debye-screened potential from perturbation theory,

$$V(r, T) \approx -\frac{g^2 C_F}{4\pi} \frac{e^{-m_D(T)r}}{r}. \quad (2)$$

However, there are a number of problems. Firstly, it is not clear if the bound state Schrödinger equation can be translated to a finite T many body situation, in a way that temperature effects show up only in the potential. Secondly, at finite T there exists a variety of non-perturbative potentials, and it is not clear which one constitutes the non-perturbative generalisation of Eq. (2).

2. Static potentials from the lattice at zero and finite T

At $T = 0$, the static potential can be defined non-perturbatively on a euclidean $L^3 \times N_\tau$ space time lattice. Consider a meson correlation function with an interpolating operator $\bar{\psi}(\mathbf{x})U(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})$, where U denotes a straight line gauge string between the quarks. In the limit $M \rightarrow \infty$ the heavy quarks can be integrated out, taking the correlator to the euclidean Wilson loop,

$$\langle \bar{\psi}(\mathbf{x}, \tau)U(\mathbf{x}, \mathbf{y}; \tau)\psi(\mathbf{y}, \tau)\bar{\psi}(\mathbf{x}, 0)U(\mathbf{x}, \mathbf{y}; 0)\psi(\mathbf{y}, 0) \rangle \longrightarrow e^{-2M\tau}W_E(|\mathbf{x} - \mathbf{y}|). \quad (3)$$

Inserting a complete set of eigenstates of the Kogut-Susskind Hamiltonian (in temporal gauge), the Wilson loop evaluates to ($r = |\mathbf{x} - \mathbf{y}|$, $U_r \equiv U(\mathbf{x}, \mathbf{y}; 0)$)

$$W_E(r, \tau) = \frac{1}{Z} \sum_{n, m} |\langle n|U_r|m \rangle|^2 e^{-E_n N_\tau} e^{-(E_m(r) - E_n)\tau} \quad (4)$$

$$\xrightarrow{N_\tau \rightarrow \infty} \sum_m |\langle 0|U_r|m \rangle|^2 e^{-(E_m(r) - E_0)\tau} \xrightarrow{\tau \rightarrow \infty} |\langle 0|U_r|1 \rangle|^2 e^{-(E_1(r) - E_0)\tau}, \quad (5)$$

with $E_m(r)$ eigenvalues in the sector with sources, and E_n in the sector without. On the lattice, $T = 1/(aN_\tau)$, hence $T = 0$ implies $N_\tau \rightarrow \infty$ in the second line. Taking furthermore the limit $\tau \rightarrow \infty$, the sum is dominated by the lowest energy state. The static potential is defined to be the lowest energy of the static quark anti-quark configuration at a given separation, $V(r) \equiv E_1(r) - E_0$. Note that the matrix element with the string operator is of no interest here.

The generalisation to finite T is difficult to interpret because of the finite and short temporal extent, $N_\tau = 1/(aT)$. Thus, we have to deal with the full superposition Eq. (4), to which now also the matrix elements contribute, and the result still depends on τ .

A different definition of the static potential which does generalise to finite T is based on the Polyakov loop $L(\mathbf{x}) = \prod_{\tau=1, N_\tau} U_0(\mathbf{x}, \tau)$, i.e. a static quark sitting at \mathbf{x} and

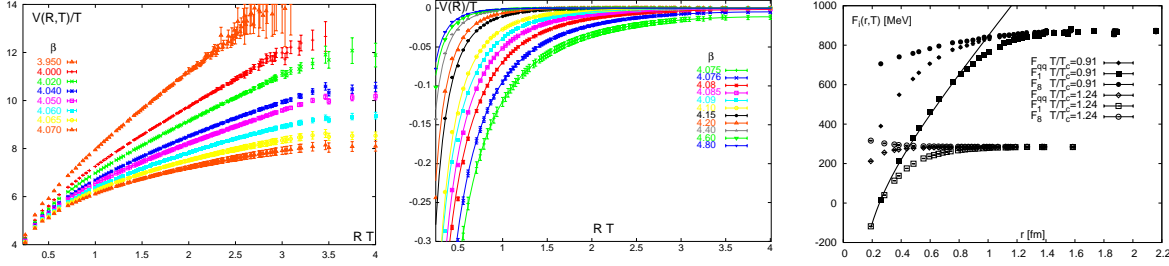


Fig. 1. Static quark anti-quark free energy/potential, Eq. (6), for $T < T_c$ (left) and $T > T_c$ (middle) [5]. Right: Free energies for the three channels Eq. (9). The solid line is the zero temperature potential [8].

propagating in euclidean time through the periodic boundary. It transforms in the adjoint, so its trace is gauge invariant. By spectral analysis one establishes that the Polyakov loop correlator represents the free energy of a static quark anti-quark pair separated by r [3],

$$e^{-F_{q\bar{q}}(r,T)/T} = \frac{1}{N_c^2} \langle \text{Tr } L^\dagger(\mathbf{x}) \text{Tr } L(\mathbf{y}) \rangle = \frac{1}{ZN_c^2} \sum_n e^{-E_n(r)/T}. \quad (6)$$

The energy levels entering this Boltzmann sum are identically the same as the $E_m(r)$ from the Wilson loop, Eq. (4). Hence, for $T \rightarrow 0$ we recover $V(r)$, cf. [4]. The free energy is thus often called a T -dependent potential, $V(r, T) \equiv F_{q\bar{q}}(r, T)$. The Polyakov loop correlator is readily simulated, with results as in Fig. 1. It gives a linear potential in the confined phase, whose string tension reduces with temperature, while in the deconfined phase the potential is screened, Fig. 1. Unfortunately, this is *not* the Debye-screened potential we want, as becomes apparent when considering its spatial decay at high T . Fitting to

$$\frac{F_{q\bar{q}}}{T} = -\frac{c(T)}{(rT)^d} e^{-m(T)r}, \quad (7)$$

gives $d \approx 1.5$ and $m = M_{0^{++}}$, i.e. the screening mass corresponds to the lightest, gauge-invariant glueball channel [6]. This can already be seen in perturbation theory, where the leading term is by two-gluon exchange and thus $m = 2m_D$ [7].

It was thus suggested to decompose the Polyakov loop correlator into channels with relative colour singlet and octet orientations of the quark anti-quark pair [3],

$$e^{-F_{q\bar{q}}(r,T)/T} = \frac{1}{9} e^{-F_1(r,T)/T} + \frac{8}{9} e^{-F_8(r,T)/T}, \quad (8)$$

$$e^{-F_1(r,T)/T} = \frac{1}{3} \langle \text{Tr } L^\dagger(\mathbf{x}) L(\mathbf{y}) \rangle,$$

$$e^{-F_8(r,T)/T} = \frac{1}{8} \langle \text{Tr } L^\dagger(\mathbf{x}) \text{Tr } L(\mathbf{y}) \rangle - \frac{1}{24} \langle \text{Tr } L^\dagger(\mathbf{x}) L(\mathbf{y}) \rangle. \quad (9)$$

Note that the correlators in the singlet and octet channels are gauge dependent, and the colour decomposition only holds perturbatively in a fixed gauge. However, in perturbation theory the singlet channel indeed displays the expected Debye-screened behaviour, $F_1(T, r) \sim e^{-m_D(T)r}/4\pi r$. This has motivated lattice simulations of these correlators in fixed Coulomb gauge, with results as in Fig. 1 (right). The three different channels

show different r -dependence, and hence lead to different binding energies when used in Schrödinger equations. There is a vast literature employing F_1 or the corresponding internal energy $U_1 = F_1 + TS_1$, and from the solutions trying to reconstruct lattice meson correlation functions to check which fits better [9].

However, both options are unphysical at a non-perturbative level. To understand this, let us start from something physical and consider a meson operator in an octet state, $O^a = \bar{\psi}(\mathbf{x})U(\mathbf{x}, \mathbf{x}_0)T^a U(\mathbf{x}_0, \mathbf{y})\psi(\mathbf{y})$, with x_0 the meson's center of mass. In the plasma the colour charge can always be neutralised by a gluon. In the correlators for the singlet and octet operators, we integrate out the heavy quarks, replacing them by Wilson lines,

$$\begin{aligned} \langle O(\mathbf{x}, \mathbf{y}; 0)O^\dagger(\mathbf{x}, \mathbf{y}; N_\tau) \rangle &\propto \langle \text{Tr } L^\dagger(\mathbf{x})U(\mathbf{x}, \mathbf{y}; 0)L(\mathbf{y})U^\dagger(\mathbf{x}, \mathbf{y}; N_\tau) \rangle, \\ \langle O^a(\mathbf{x}, \mathbf{y}; 0)O^{a\dagger}(\mathbf{x}, \mathbf{y}; N_\tau) \rangle &\propto \left[\frac{1}{N_c^2 - 1} \langle \text{Tr } L^\dagger(\mathbf{x}) \text{Tr } L(\mathbf{y}) \rangle \right. \\ &\quad \left. - \frac{1}{N_c(N_c^2 - 1)} \langle \text{Tr } L^\dagger(\mathbf{x})U(\mathbf{x}, \mathbf{y}; 0)L(\mathbf{y})U^\dagger(\mathbf{x}, \mathbf{y}; N_\tau) \rangle \right]. \end{aligned} \quad (10)$$

We have now arrived at gauge invariant expressions, because we used a gauge string between the sources. The singlet correlator corresponds to a periodic Wilson loop which wraps around the boundary. The connection to the gauge fixed correlators is readily established, replacing the gauge string by gauge fixing functions, $U(\mathbf{x}, \mathbf{y}) = g^{-1}(\mathbf{x})g(\mathbf{y})$. Thus, in axial gauge $U(\mathbf{x}, \mathbf{y}) = 1$ (and only there) the gauge fixed correlators are identical to the gauge invariant ones.

Next, let us perform the spectral analysis. While indeed the energy eigenvalues in the spectral sum are independent of the operators [10], the full correlators take the form [11]

$$\begin{aligned} e^{-F_1(r, T)/T} &= \frac{1}{ZN_c^2} \sum_n \langle n_{\delta\gamma} | U_{\gamma\delta}(\mathbf{x}, \mathbf{y}) U_{\alpha\beta}^\dagger(\mathbf{x}, \mathbf{y}) | n_{\beta\alpha} \rangle e^{-E_n(r)/T}, \\ e^{-F_8(r, T)/T} &= \frac{1}{ZN_c^2} \sum_n \langle n_{\delta\gamma} | U_{\gamma\delta}^a(\mathbf{x}, \mathbf{y}) U_{\alpha\beta}^{a\dagger}(\mathbf{x}, \mathbf{y}) | n_{\beta\alpha} \rangle e^{-E_n(r)/T}. \end{aligned} \quad (11)$$

The energy levels in the exponents are identically the same in Eqs. (6,11) and correspond to the familiar gauge invariant static potential at zero temperature and its excitations. However, while Eq. (6) is purely a sum of exponentials and thus a true free energy, the singlet and octet correlators contain matrix elements which *do* depend on the operators used, thus giving a path/gauge dependent weight to the exponentials contributing to F_1, F_8 . This is illustrated numerically in Fig. 2 in the low temperature limit, where the ground state potential dominates and one can cleanly separate the exponential and the matrix elements. The r -dependent structure is entirely in the matrix elements, which depend on operators and/or the gauge.

I do not see how this is evaded by applying smearing techniques, as recently suggested in [12]. These authors replace the spatial string with a smeared object in order to increase the overlap with the ground state, i.e. to get the ground state matrix element close to one. However, most smearing changes the expectation values of correlators, thus destroying their mutual relations, Eq. (9). Secondly, smearing increases the weight of the lower energy states at the cost of the higher ones, and thus undoes the effect of finite temperature in a procedure dependent way. Finally, if one could get all matrix

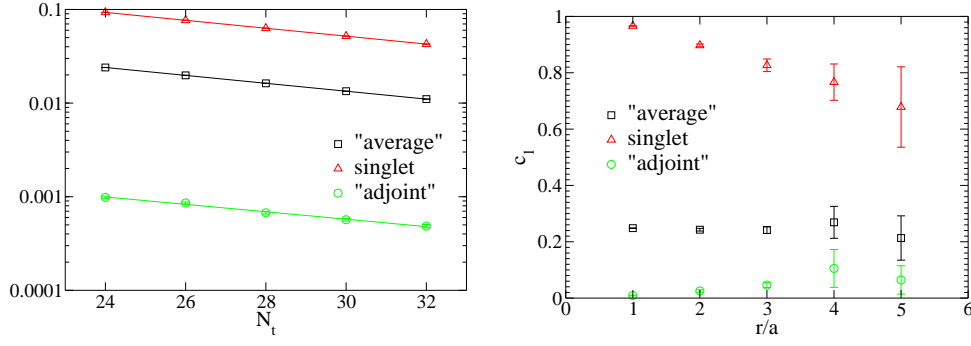


Fig. 2. Left: Polyakov loop correlators, Eq. (9), for $r/a = 1$ in the case of 3d SU(2) in the low temperature limit. All decay with the same ground state exponential. Right: The corresponding matrix elements in the three channels introduces operator dependent r -dependence, except for the average channel [11].

elements equal to one, the different channels would simply be equal, up to the trivial colour coefficients, with no additional information.

To summarise, since the spectral information contained in the average and gauge fixed singlet and octet channels is the same, we must conclude that any difference between those correlators is entirely gauge dependent and thus unphysical, and so are all binding energies calculated in F_1 or U_1 .

3. A real time static potential for finite T quarkonia

Progress was made recently by generalising the effective theory approach quarkonium physics at $T = 0$, namely pNRQCD, to finite temperatures [13,14,15], as reviewed at this conference [16]. The analysis is performed in a perturbative setting in Minkowski time. Just as at zero temperature, the static potential then appears as a matching coefficient in the effective theory after the heavy modes have been integrated out. The relevant correlation function is the quarkonium correlator in real time, but evaluated as a thermal expectation value. Not surprisingly, after integrating out the static quarks, the correlator is proportional to a Wilson loop in Minkowski time, $W_E(it, \mathbf{r})$. Of course, the expectation value implied in W_E is now a thermal one, i.e. N_τ is finite for fixed lattice spacing. Hence we need the analytic continuation of the double spectral sum in Eq. (4). From the effective theory it is easy to see that this correlator obeys a real time evolution equation

$$[i\partial_t - V_>(t, r)]W_E(it, r) = 0. \quad (12)$$

This represents the desired Schrödinger equation for quarkonia in the plasma, and defines the relevant real time dependent potential. The required scale hierarchy for this equation to be valid is $g^2M < T < gM$. Furthermore, for non-relativistic bound states $p \ll E$, hence we need $t \gg r$, i.e. the static potential is obtained in the long time limit $V(\infty, r)$.

Eq. (12) may be also be viewed as a non-perturbative definition of the potential of interest via a correlation function, just as was the case for the zero temperature potential. Unfortunately, this one is defined for Minkowski time and thus requires analytic continuation, i.e. it cannot be evaluated directly from euclidean lattice simulations. However, a first impression about this object can be gained from HTL-resummed perturbation theory, for which the leading order result is

$$V_{>}(\infty, r) = -\frac{g^2 C_F}{4\pi} \left[m_D + \frac{\exp(-m_D r)}{r} \right] - \frac{ig^2 T C_F}{4\pi} \phi(m_D r),$$

with $\phi(x) = 2 \int_0^\infty \frac{dz z}{(z^2 + 1)^2} \left[1 - \frac{\sin(zx)}{zx} \right].$ (13)

The most striking feature of this potential is that it is complex, contrary to the free energies discussed before. The real part features the expected Debye-screened potential. The imaginary part is due to Landau damping and must necessarily be there for a correct effective description of the plasma dynamics. Its derivation and properties are discussed in more detail in [13,14,16].

What are the corrections to this potential? Firstly, there are corrections from HTL-resummed perturbation theory of the order $g^2 T/\Lambda$, where Λ is the UV cut-off, with a calculable coefficient. Here, we are interested in the non-perturbative corrections from infrared modes $\sim g^2 T/m_{mag}$. These are due to the soft colour magnetic modes $m_{mag} \sim g^2 T$, and thus cannot be calculated in perturbation theory.

However, one can calculate these non-perturbative corrections by classical lattice simulations in that sector of the theory, which has high occupation numbers and is well represented by a classical approximation. To identify this sector it is instructive to take the limit $\hbar \rightarrow 0$ in our perturbative result Eq. (13) first. To this end, \hbar needs to be reinstated by the replacements $g^2 \rightarrow g^2 \hbar$, $1/T \rightarrow \hbar/T$, leading to

$$\lim_{\hbar \rightarrow 0} V_{>}(\infty, r) = -\frac{ig^2 T C_F}{4\pi} \phi(m_D r). \quad (14)$$

Thus, only the imaginary part survives in the continuum limit. This is easy to understand since the long range physics of Landau damping is dominated by classical fields, e.g. in scalar field dynamics, whereas the binding is a generic quantum effect, cf. the hydrogen problem. Thus, we can evaluate the non-perturbative infrared effects for the imaginary part of the potential.

4. Imaginary part from classical lattice simulations

This has been done in [17], following the technical setup that was also used for the evaluation of the sphaleron rate in the electroweak theory [18]. In order to perform real time simulations one reformulates the theory in a Hamiltonian approach. Fixing temporal gauge $U_0 = 0$, the conjugate field operators are the links and the electric fields defined by $\dot{U}(\mathbf{x}, t) = iE_i(\mathbf{x}, t)U_i(\mathbf{x}, t)$. Full gauge invariance is restored by imposing the Gauss constraint $G(x) \equiv \sum_i \left[E_i(x) - U_{-i}(x)E_i(x - \hat{i})U_{-i}^\dagger(x) \right] - j^0(x) = 0$. A thermal distribution at some initial time is generated by the partition function

$$Z = \int \mathcal{D}U_i \mathcal{D}E_i \delta(G) e^{-\beta H}, \quad H = \frac{1}{N_c} \sum_x \left[\sum_{i < j} \text{Re Tr}(1 - U_{ij}) + \frac{1}{2} \text{Tr}(E_i^2) \right], \quad (15)$$

where U_{ij} is the plaquette. The distribution is then evolved in real time by the classical equations of motion,

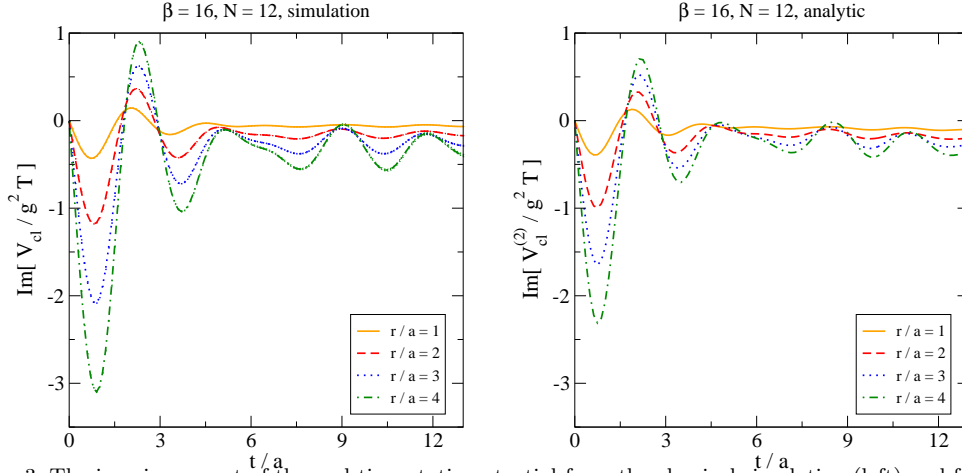


Fig. 3. The imaginary part of the real-time static potential from the classical simulation (left) and from resummed perturbation theory (right)[17].

$$\dot{U}_i(x) = iE_i(x)U_i(x), \quad E_i = \sum_a E_i^a T^a, \quad \dot{E}_i^a(x) = -2 \text{Im Tr}[T^a \sum_{|j| \neq i} U_{ij}(x)]. \quad (16)$$

This procedure can be improved by taking into account quantum corrections by using a HTL-resummed effective theory. It will generate a source term due to the hard particles in the plasma, which modifies the Hamiltonian as well as the equations of motion by coupling the classical fields to the quantum effects of the hard particles. That approach was used in [18] and is easily adapted to the present problem [17].

One can now calculate the classical thermal expectation value of the real time Wilson loop, which in temporal gauge reduces to a correlation function of the spatial string,

$$W_{E,cl}(it, r) = \langle U^\dagger(\mathbf{x}, \mathbf{y}; t) U(\mathbf{x}, \mathbf{y}; 0) \rangle. \quad (17)$$

The time dependent potential extracted via Eq. (12) is shown in Fig. 3, where it is compared with the result obtained from resummed classical lattice perturbation theory. We observe complete qualitative agreement. Note in particular that a finite imaginary part survives also non-perturbatively in the long time limit, when fluctuations have died out. Comparing the values one finds that the non-perturbative effects make the imaginary part more negative, i.e. increase Landau damping.

5. Non-perturbative real time potential?

It is clearly desirable to go beyond the classical approximation and construct an operator from which the whole quantum potential, including its real part, can be extracted. Of course, a full quantum computation of the real time Wilson loop is impossible, just as the calculation of any other real time correlation function. The whole point of the potential approach is to bypass the need for such correlators. In our case, what is needed is the static potential in the infinite time limit, which is clearly less information than having to know the full time dependence. In terms of correlators, the information we need is

$$\lim_{t \rightarrow \infty} W_E(it, r), \quad \lim_{t \rightarrow \infty} \partial_t W_E(it, r). \quad (18)$$

At least in principle, these limits ought to be representable by Euclidean operators, the challenge is to construct those in practice.

6. Conclusions

We have argued that many potential models used for the description of quarkonia at finite T have significant flaws. The connection between the Schrödinger equation to the underlying quantum field theory is unclear, and lattice potentials extracted from Polyakov loop correlators, which are typically used as input for those models, are the wrong quantities for this purpose. The average free energy is gauge invariant and well defined, but in its perturbative limit does not reduce to the Debye-screened potential. The so-called singlet potential is gauge dependent and therefore unphysical.

These problems can be overcome by using an effective field theory obtained by integrating out the heavy quarks, which is pNRQCD generalised to finite temperatures. The resulting Schrödinger equation is the real time evolution equation for a quarkonium correlator, and the static potential in this equation is a matching coefficient in the effective theory. To leading order in HTL-resummed perturbation theory, this potential is complex, its real part showing the correct Debye-screened behaviour and its imaginary part reflecting Landau damping. In the classical limit, only the imaginary part survives. This part can be calculated non-perturbatively with classical lattice simulations in real time. The result agrees in all qualitative features with the HTL result with slightly strengthened damping. It is now important to search for a lattice operator that represents the real part.

Acknowledgements: Some of the work presented here is supported by the BMBF project *Hot Nuclear Matter from Heavy Ion Collisions and its Understanding from QCD*.

References

- [1] T. Matsui and H. Satz, Phys. Lett. B **178** (1986) 416.
- [2] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B **566** (2000) 275.
- [3] L. D. McLerran and B. Svetitsky, Phys. Rev. D **24** (1981) 450.
- [4] M. Lüscher and P. Weisz, JHEP **0207** (2002) 049 and references therein.
- [5] O. Kaczmarek, F. Karsch, E. Laermann and M. Lutgemeier, Phys. Rev. D **62** (2000) 034021.
- [6] A. Hart, M. Laine and O. Philipsen, Nucl. Phys. B **586** (2000) 443,
S. Datta and S. Gupta, Phys. Rev. D **67** (2003) 054503.
- [7] S. Nadkarni, Phys. Rev. D **33** (1986) 3738; Phys. Rev. D **34** (1986) 3904.
- [8] O. Kaczmarek and F. Zantow, Phys. Rev. D **71** (2005) 114510.
- [9] A. Mocsy and P. Petreczky, Phys. Rev. D **77** (2008) 014501.
- [10] O. Philipsen, Phys. Lett. B **535** (2002) 138.
- [11] O. Jahn and O. Philipsen, Phys. Rev. D **70** (2004) 074504.
- [12] A. Bazavov, P. Petreczky and A. Velytsky, arXiv:0809.2062 [hep-lat].
- [13] M. Laine, O. Philipsen, P. Romatschke and M. Tassler, JHEP **0703** (2007) 054.
- [14] N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky, Phys. Rev. D **78** (2008) 014017.
- [15] M. A. Escobedo and J. Soto, arXiv:0804.0691 [hep-ph].
- [16] M. Laine, arXiv:0810.1112 [hep-ph].
- [17] M. Laine, O. Philipsen and M. Tassler, JHEP **0709** (2007) 066.
- [18] D. Bödeker, G. D. Moore and K. Rummukainen, Phys. Rev. D **61** (2000) 056003.